ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 352 (2009) 99–111

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On a Sobolev-type inequality related to the weighted p-Laplace operator ☆

Marita Gazzini [a], Roberta Musina [b,*]

[a] *SISSA, Via Beirut 4, 34010 Trieste, Italy*

[b] *Dipartimento di Matematica ed Informatica, Università di Udine, via delle Scienze 206, 33100 Udine, Italy*

Received 8 January 2008

Available online 20 June 2008

Submitted by V. Radulescu

Abstract

In this paper we deal with some Sobolev-type inequalities with weights that were proved by Maz'ya in [V.G. Maz'ja, Sobolev Spaces, Springer-Verlag, Berlin, 1980] and by Caffarelli, Kohn and Nirenberg in [L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weight, Compos. Math. 53 (1984) 259–275]. For integers $1 \leqslant k \leqslant N$ denote points $\xi \in \mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$ as pairs (x, y). Let $p \in (1, N)$, $q \in (p, p^*]$ and assume $b_a := N - q\frac{N-p+a}{p} < k$. Then there exists $c > 0$ such that

$$c\left(\int_{\mathbb{R}^N} |x|^{-b_a}|u|^q \, d\xi\right)^{p/q} \leqslant \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, d\xi, \quad \forall u \in C_c^\infty(\mathbb{R}^N).$$

We prove that the best constant is achieved for any a, p, k, provided that $q < p^*$ or $q = p^*$ and $a < 0$. Results for weighted Sobolev-type inequalities on cones are also given.

Keywords: Variational methods; Critical growth; Weighted L^p-Laplace operator; Caffarelli–Kohn–Nirenberg inequalities; Maz'ya inequalities

0. Introduction

Let k, N be positive integers with $1 \leqslant k \leqslant N$. We put $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, and we denote points ξ in $\mathbb{R}^N$ as pairs $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Let a, p, q be real parameters, such that

$$1 < p < N, \qquad (p-N)\frac{k}{N} < a, \qquad \max\left\{p, \frac{p(N-k)}{N-p+a}\right\} < q \leqslant p^* = \frac{pN}{N-p}, \tag{0.1}$$

☆ Work supported by Cofin. MIUR Progetto di Ricerca "Metodi Variazionali ed Equazioni Differenziali Nonlineari."

* Corresponding author.

E-mail addresses: gazzini@sissa.it (M. Gazzini), musina@dimi.uniud.it (R. Musina).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.06.021

and set

$$b_a = b_a(p,q) := N - q\frac{N-p+a}{p}. \tag{0.2}$$

Then there exists a constant $c > 0$ such that

$$c\left(\int_{\mathbb{R}^N} |x|^{-b_a}|u|^q\,d\xi\right)^{p/q} \leqslant \int_{\mathbb{R}^N} |x|^a|\nabla u|^p\,d\xi, \quad \forall u \in C_c^\infty(\mathbb{R}^N). \tag{0.3}$$

Inequality (0.3) was proved by Caffarelli, Kohn and Nirenberg in [11] for spherical weights, and by Maz'ya, in Section 2.1.6 of [27], in the cylindrical case $k < N$. In this paper we investigate the natural question of the existence of extremals for the best constant

$$S_{a,q}(p) := \inf_{\mathcal{D}^{1,p}(\mathbb{R}^N;|x|^a\,d\xi)} \frac{\int_{\mathbb{R}^N}|x|^a|\nabla u|^p\,d\xi}{(\int_{\mathbb{R}^N}|x|^{-b_a}|u|^q\,d\xi)^{p/q}}. \tag{0.4}$$

Here the Banach space $\mathcal{D}^{1,p}(\mathbb{R}^N;|x|^a\,d\xi)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to $\|u\| := (\int_{\mathbb{R}^N}|x|^a|\nabla u|^p)^{1/p}$, which turns out to be a norm by (0.3). Its elements can be identified as measurable functions (modulo a.e.), and it is continuously imbedded into $L^q(\mathbb{R}^N;|x|^{-b_a}\,d\xi)$ by (0.3).

Extremals for $S_{a,q}(p)$ are ground state solutions of the Euler–Lagrange equation

$$-\operatorname{div}\big(|x|^a|\nabla u|^{p-2}\nabla u\big) = |x|^{-b_a}|u|^{q-2}u \quad \text{on } \mathbb{R}^N, \tag{0.5}$$

which can be regarded as a model for more general degenerate and singular elliptic equations.

Several existence result are available in the literature. For $a = 0$ and $q = p^*$ the infimum $S_{a,q}(p)$ coincides with the Sobolev constant $S(p)$. It is achieved on $\mathcal{D}^{1,p}(\mathbb{R}^N)$ by an explicitly known radially symmetric map (see [31]). In case $k = N$, Catrina and Wang have proved existence for $p = 2$, $q < 2^*$ or $q = 2^*$ and $a < 0$ ([14], see also [23]). A related critical problem is studied in the recent paper [17]. Finally, as concerns the cylindrical case $k < N$ we quote [6], where $a = 0$, $k \geqslant p$ and $q \in (p, p^*)$ are assumed, and [28,32], that deal with $p = 2$, $a \geqslant 2 - k$.

Our approach to the minimization problem (0.4) looks quite simple and flexible, and it is uniform with respect to the parameters k, N, a, p and q. Our first main theorem is the following.

Theorem 0.1. *Assume that* (0.1) *is satisfied. Then $S_{a,p^*}(p)$ is achieved provided that*

$$q < p^* \quad \text{or} \quad q = p^* \quad \text{and} \quad S_{a,p^*}(p) < S(p).$$

As one can expect, the limiting case $q = p^*$ is more difficult. It will be shown in Proposition A.8 that $S_{a,p^*}(p) \leqslant S(p)$ for any exponent a. The strict inequality holds true whenever a is negative (see also Proposition A.10 for $p = 2$, $k = 1$). This is our second main result.

Theorem 0.2. *Let $p \in (1, N)$. Then $S_{a,p^*}(p)$ is achieved provided that $(p-N)\frac{k}{N} < a < 0$.*

Theorem 0.1 extends the existence result in [6] to $a = 0$, $k < p$ and $\frac{p(N-k)}{N-p} < q < p^*$. Finally, we notice that Theorem 0.2 gives a positive answer to a question that has been raised by Tertikas and Tintarev in [32, Section 6, at point 4], at least when $p < k$.

Let us describe the main features of problem (0.3) and our approach. While studying (0.4) one has to take into account the action of the groups of dilations in $\mathbb{R}^N$. If $k < N$ also the group of translations in $\mathbb{R}^{N-k}$ has to be considered. Indeed, for any minimizing sequence u_h, and for arbitrary sequences $t_h \in (0, +\infty)$, $y_h \in \mathbb{R}^{N-k}$, it turns out that $\tilde{u}_h(x, y) := u_h(t_h x, t_h y + y_h)$ still approaches the infimum $S_{a,q}(p)$. By this remark it is quite easy to exhibit noncompact minimizing sequences. The group of translations in the x-variable produces worse lack of compactness phenomena in the limiting case $q = p^*$, since minimizing sequences for (0.4) might blow-up an extremal for the Sobolev constant $S(p)$.

Our main idea to prove Theorem 0.1 simply consists in looking for a "good" minimizing sequence. This strategy has been already followed in [28]. The main tools are the Ekeland variational principle, a Rellich-type theorem, and a suitable rescaling argument. In particular, we skip the blow-up analysis of all minimizing sequences (as it has been done for example in [6,14,32]). We do not require the Brezis–Lieb Lemma, the Concentration–Compactness Lemmata by P.L. Lions, as well as any similar result.

The crucial step is Proposition 1.3. It deals with the asymptotic behavior of bounded sequences of approximate solutions to (0.5). Thanks to Proposition 1.3, we can find a weakly convergent minimizing sequence u_h whose L^q-norms are bounded away from 0 on a compact subset of $(\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$. In this way we exclude concentration at 0 and vanishing, and we overcome the lack of compactness produced by dilations in $\mathbb{R}^N$ and by translations in $\mathbb{R}^{N-k}$. If $q < p$ we use the Rellich Theorem and we conclude that u_h converges weakly to some $u \neq 0$. Then standard arguments imply that u achieves $S_{a,q}(p)$. If $q = p^*$ the assumption $S_{a,p^*}(p) < S(p)$ prevents concentration phenomena at points (x_0, y_0) with $x_0 \neq 0$, and we can conclude as in the subcritical case.

There is a large number of papers that are related to (0.3). Besides the above quoted papers we cite for example [1–5,7,9,12,16,18,19,21,22,29,30,34] for $k = N$, and [8,13,20,24–26,33] for $k < N$.

The paper is organized as follows:

- in Sections 1 and 2 we prove our main theorems;
- in Section 3 we establish a suitable Sobolev-type inequality on cones for any $a \in \mathbb{R}$, $a \neq p - k$ (compare with Lemma 3.1), and we discuss the existence of extremal functions. The main result is stated in Theorem 3.2;
- in Appendix A we collect additional results, remarks and open problems. In particular we draw here our attention on the differential equation (0.5) and on the limiting case $q = p^*$.

Notation. For any integer $j \geqslant 1$, we denote by $B^j_R(z)$ the j-dimensional ball of radius R centered at $z \in \mathbb{R}^j$. The Lebesgue measure of a domain Ω in $\mathbb{R}^N$ is denoted by $|\Omega|$.

Let $q \in [1, +\infty)$, $\alpha \in \mathbb{R}$, and let Ω be a domain in $\mathbb{R}^N$. Then $L^q(\Omega; |x|^\alpha\, d\xi)$ is the space of measurable maps u on Ω with $\int_\Omega |x|^\alpha |u|^q\, d\xi < +\infty$, so that $L^q(\Omega; |x|^0\, d\xi) \equiv L^q(\Omega)$ is the standard Lebesgue space. For $p > 1$ the reflexive Banach space $\mathcal{D}^{1,p}(\Omega)$ is the closure of $C^\infty_c(\Omega)$ with respect to the L^p-norm of ∇u. The Sobolev critical exponent is $p^* = \frac{Np}{N-p}$. We recall that the best constant

$$S(p) := S_{0,p^*}(p) = \inf_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^p\, d\xi}{(\int_{\mathbb{R}^N} |u|^{p^*}\, d\xi)^{p/p^*}}$$

is achieved on $\mathcal{D}^{1,p}(\mathbb{R}^N)$ by the map

$$U(\xi) := \bigl(1 + |\xi|^{\frac{p}{p-1}}\bigr)^{-\frac{N-p}{p}}.$$

1. Proof of Theorem 0.1

In this proof we focus our attention on the more delicate case $k < N$. When $|x|$ is the distance from the origin, that is $k = N$, the argument is a little bit simpler, as the problem has less invariances.

We start with two technical lemmata. The first one is a Rellich-type result.

Lemma 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, d\xi) \hookrightarrow L^p(\Omega, |x|^a\, d\xi)$ with compact inclusion.*

Proof. Fix a map $u \in C^\infty_c(\mathbb{R}^N)$. Hölder inequality and (0.3) give

$$\int_\Omega |x|^a |u|^p\, d\xi \leqslant |\Omega|^{\frac{p}{N}} \biggl(\int_\Omega |x|^{\frac{Na}{N-p}} |u|^{p^*}\, d\xi \biggr)^{p/p^*} \leqslant c|\Omega|^{\frac{p}{N}} \int_{\mathbb{R}^N} |x|^a |\nabla u|^p\, d\xi, \tag{1.1}$$

where c does not depend on u. This proves the continuity of the embedding. To prove compactness take a sequence u_h in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, d\xi)$, with $u_h \to 0$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, d\xi)$. Fix $\varepsilon > 0$ and take a smooth function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^k)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi_\varepsilon(x) = 0$ if $|x| \leqslant \varepsilon^2$, and $\varphi_\varepsilon(x) = 1$ if $|x| \geqslant \varepsilon$. By Rellich Theorem, it turns out that

$$\int_\Omega |x|^a |\varphi_\varepsilon u_h|^p\, d\xi = o(1)$$

as $h \to +\infty$, since $|x|$ is bounded away from 0 on the support of φ_ε. On the other hand, the sequence u_h is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$, and therefore from (1.1) one gets

$$\int_\Omega |x|^a \big|(1-\varphi_\varepsilon)u_h\big|^p\,d\xi \leqslant c|\Omega_\varepsilon|^{\frac{p}{N}} \int_{\mathbb{R}^N} |x|^a |\nabla u_h|^p\,d\xi \leqslant c|\Omega_\varepsilon|^{\frac{p}{N}},$$

where $\Omega_\varepsilon := \{(x,y) \in \Omega \mid |x| < \varepsilon\}$. Writing $u_h = \varphi_\varepsilon u_h + (1-\varphi_\varepsilon)u_h$ one infers that

$$\int_\Omega |x|^a |u_h|^p\,d\xi \leqslant c \int_\Omega |x|^a \big(|\varphi_\varepsilon u_h|^p + \big|(1-\varphi_\varepsilon)u_h\big|^p\big)\,d\xi \leqslant o(1) + c|\Omega_\varepsilon|^{\frac{p}{N}}$$

for ε fixed, as $h \to +\infty$. The conclusion easily follows, since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$. □

Lemma 1.2. *If $\Psi \in C_c^\infty(\mathbb{R}^N)$ then $\Psi u \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$ for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$.*

Proof. We can approximate any fixed $u \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$ with a sequence $u_h \in C_c^\infty(\mathbb{R}^N)$. Using Lemma 1.1 it is easy to prove that $\Psi u_h \to \Psi u$ in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$, hence $\Psi u \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$. □

The main step in the proof of Theorem 0.1 is the following proposition, that is concerned with approximate solutions to (0.5).

Proposition 1.3. *Let u_h be a bounded sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$, and let $f_h \to 0$ be a sequence in the dual of $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$. Assume that for a, p, q, b_a as in* (0.1)*,* (0.2) *it holds that*

$$-\operatorname{div}\big(|x|^a |\nabla u_h|^{p-2}\nabla u_h\big) = |x|^{-b_a}|u_h|^{q-2}u_h + f_h.$$

Then, up to a subsequence, either $u_h \to 0$ strongly in $L^q(\mathbb{R}^N; |x|^{-b_a}\,d\xi)$, or there exist sequences $(t_h)_h \subset (0,+\infty)$ and $(\eta_h)_h \subset \mathbb{R}^{N-k}$, such that

$$\lim_{h\to+\infty} \int_K |x|^{-b_a}|\tilde u_h|^q\,d\xi > 0,$$

where $\tilde u_h(x,y) = t_h^{\frac{N-p+a}{p}} u_h(t_h x, t_h y + \eta_h)$ and $K = \{(x,y) \in \mathbb{R}^N \mid \frac{1}{2} < |x| < 1, |y| < 1\}$.

Proof. We can assume that there exists $u \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$ such that $u_h \to u$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$ and in $L^q(\mathbb{R}^N; |x|^{-b_a}\,d\xi)$. If $u \neq 0$ then we are done since, up to a rescaling, $\int_K |x|^{-b_a}|u|^q\,d\xi > 0$. Then the conclusion follows by the weak lower semicontinuity of the L^q-norm. Therefore, we assume

$$u_h \to 0 \quad \text{weakly in } \mathcal{D}^{1,p}\big(\mathbb{R}^N; |x|^a\,d\xi\big) \quad \text{and} \quad \lim_{h\to+\infty} \int_{\mathbb{R}^N} |x|^{-b_a}|u_h|^q\,d\xi > 0.$$

Fix $\varepsilon_0 > 0$ in such a way that

$$\varepsilon_0^{\frac{q}{q-p}} < \lim_{h\to+\infty} \int_{\mathbb{R}^N} |x|^{-b_a}|u_h|^q\,d\xi, \quad 2\varepsilon_0 < S_{a,q}(p).$$

Using in a standard way the concentration function

$$Q_h(t) := \sup_{\eta\in\mathbb{R}^{N-k}} \int_{B_t^k(0)\times B_t^{N-k}(\eta)} |x|^{-b_a}|u_h|^q\,d\xi,$$

it is possible to select $t_h > 0$ and $\eta_h \in \mathbb{R}^{N-k}$ such that the rescaled sequence

$$\tilde u_h(x,y) := t_h^{\frac{N-p+a}{p}} u_h(t_h x, t_h y + \eta_h)$$

satisfies $\int |x|^a |\nabla \tilde{u}_h|^p = \int |x|^a |\nabla u_h|^p = O(1)$, and

$$\int\limits_{B_1^k(0)\times B_1^{N-k}(y)} |x|^{-b_a} |\tilde{u}_h|^q \, d\xi \leqslant (2\varepsilon_0)^{\frac{q}{q-p}} \quad \forall y \in \mathbb{R}^{N-k}, \tag{1.2}$$

$$\int\limits_{B_1^k(0)\times B_1^{N-k}(0)} |x|^{-b_a} |\tilde{u}_h|^q \, d\xi \geqslant \varepsilon_0^{\frac{q}{q-p}} > 0, \tag{1.3}$$

$$-\operatorname{div}\big(|x|^a |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h\big) = |x|^{-b_a} |\tilde{u}_h|^{q-2} \tilde{u}_h + \tilde{f}_h, \tag{1.4}$$

with $\tilde{f}_h \to 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a \, d\xi)'$. As before, if (up to a subsequence) $\tilde{u}_h \to \tilde{u} \neq 0$ then we are done. If $\tilde{u}_h \to 0$, choose a finite number of points $y_1, \dots, y_s \in \mathbb{R}^{N-k}$ such that

$$\overline{B_1^{N-k}(0)} \subset \bigcup_{j=1}^{s} B_{1/2}^{N-k}(y_j). \tag{1.5}$$

Let $\psi_1, \dots, \psi_s$ be cut-off functions, with $\psi_j = \psi_j(y) \in C_c^\infty(B_1^{N-k}(y_j))$, $\psi_j \equiv 1$ on $B_{1/2}^{N-k}(y_j)$ and $0 \leqslant \psi_j \leqslant 1$. Also, fix a map $\varphi = \varphi(x) \in C_c^\infty(B_1^k(0))$ satisfying $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on $B_{1/2}^k(0)$. Thanks to Lemma 1.2 we can use $\varphi^p \psi_i^p \tilde{u}_h$ as test function in (1.4) to find

$$\int\limits_{\mathbb{R}^N} |x|^a |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h \cdot \nabla\big(\varphi^p \psi_j^p \tilde{u}_h\big) \, d\xi = \int\limits_{\mathbb{R}^N} |x|^{-b_a} |\tilde{u}_h|^{q-p} |\varphi \psi_j \tilde{u}_h|^p \, d\xi + o(1). \tag{1.6}$$

Direct computations and Lemma 1.1 give

$$\int\limits_{\mathbb{R}^N} |x|^a |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h \cdot \nabla\big(\varphi^p \psi_j^p \tilde{u}_h\big) \, d\xi = \int\limits_{\mathbb{R}^N} |x|^a \big|\nabla(\varphi \psi \tilde{u}_h)\big|^p \, d\xi + o(1).$$

Thus, we can use Hölder inequality, (1.2), (1.6) and the definition of $S_{a,q}(p)$ to infer that

$$S_{a,q}(p)\Bigg(\int\limits_{\mathbb{R}^N} |x|^{-b_a} |\varphi \psi_j \tilde{u}_h|^q \, d\xi\Bigg)^{\frac{p}{q}} \leqslant 2\varepsilon_0 \Bigg(\int\limits_{\mathbb{R}^N} |x|^{-b_a} |\varphi \psi_j \tilde{u}_h|^q \, d\xi\Bigg)^{\frac{p}{q}} + o(1).$$

Since $2\varepsilon_0 < S_{a,q}(p)$ this implies that $\int_{\mathbb{R}^N} |x|^{-b_a} |\varphi \psi_j \tilde{u}_h|^q \, d\xi = o(1)$, and therefore, by (1.5),

$$\int\limits_{B_{1/2}^k(0)\times B_1^{N-k}(0)} |x|^{-b_a} |\tilde{u}_h|^q \, d\xi \leqslant \sum_{j=1}^{s} \int\limits_{B_{1/2}^k(0)\times B_{1/2}^{N-k}(y_j)} |x|^{-b_a} |\tilde{u}_h|^q \, d\xi = o(1).$$

Finally, from (1.3) we get

$$0 < \varepsilon_0^{\frac{q}{q-p}} < \int\limits_{B_1^k(0)\times B_1^{N-k}(0)} |x|^{-b_a} |\tilde{u}_h|^q \, d\xi = \int\limits_{K} |x|^{-b_a} |\tilde{u}_h|^q \, d\xi + o(1).$$

Proposition 1.3 is completely proved. □

Proof of Theorem 0.1. Take a minimizing sequence u_h satisfying

$$\int\limits_{\mathbb{R}^N} |x|^{-b_a} |u_h|^q \, d\xi = \big(S_{a,q}(p)\big)^{\frac{q}{q-p}}, \qquad \int\limits_{\mathbb{R}^N} |x|^a |\nabla u_h|^p \, d\xi = \big(S_{a,q}(p)\big)^{\frac{q}{q-p}} + o(1). \tag{1.7}$$

By Ekeland's variational principle, we can assume that

$$-\operatorname{div}\big(|x|^a |\nabla u_h|^{p-2} \nabla u_h\big) = |x|^{-b_a} |u_h|^{q-2} u_h + f_h, \tag{1.8}$$

where $f_h \to 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)'$. Up to a subsequence, we can find $u \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$ such that $u_h \to u$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$. Thanks to Proposition 1.3 we can assume that, up to a change of coordinates,

$$\lim_{h\to+\infty} \int\limits_K |x|^{-b_a}|u_h|^q\,d\xi > 0, \tag{1.9}$$

where $K = \{(x,y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \mid \frac{1}{2} < |x| < 1,\ |y| < 1\}$. We claim that $u \neq 0$. This is immediate if $q < p^*$, since in this case $\int_K |x|^{-b_a}|u|^q\,d\xi = \lim_{h\to+\infty} \int_K |x|^{-b_a}|u_h|^q\,d\xi > 0$ by Rellich Theorem. Therefore we take $q = p^*$. By contradiction, assume that $u_h \to u$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$. Choose smooth maps $\varphi \in C_c^\infty(\mathbb{R}^k)$ and $\psi \in C_c^\infty(\mathbb{R}^{N-k})$ in such a way that $\varphi(x) = 0$ for $|x| \leqslant \frac{1}{4}$, $\varphi(x) = 1$ for $\frac{1}{2} \leqslant |x| \leqslant 1$ and $\psi(y) = 1$ for $|y| \leqslant 1$. Notice that $\varphi\psi \equiv 1$ on K. Since $\langle f_h, \varphi^p\psi^p u_h\rangle = o(1)$, we can argue as in the proof of Proposition 1.3 to get

$$\int\limits_{\mathbb{R}^N} |x|^a \big|\nabla(\varphi\psi u_h)\big|^p d\xi \leqslant S_{a,p^*}(p)\Bigg(\int\limits_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*}\,d\xi\Bigg)^{\frac{p}{p^*}} + o(1). \tag{1.10}$$

Now, notice that $|x|^{\frac{a}{p}}\nabla(\varphi\psi u_h) = \nabla(|x|^{\frac{a}{p}}\varphi\psi u_h) - F_h$, where $F_h := \varphi\psi u_h \nabla(|x|^{\frac{a}{p}})$. Since $\varphi\psi$ has compact support and since it vanishes in a neighborhood of the singular set $\{x = 0\}$, then $F_h \to 0$ in $L^p(\mathbb{R}^N)^N$ by Rellich Theorem. Therefore

$$\int\limits_{\mathbb{R}^N} |x|^a \big|\nabla(\varphi\psi u_h)\big|^p\,d\xi = \int\limits_{\mathbb{R}^N} \bigg|\nabla\Big(|x|^{\frac{a}{p}}\varphi\psi u_h\Big)\bigg|^p d\xi + o(1) \geqslant S(p)\Bigg(\int\limits_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*}\,d\xi\Bigg)^{\frac{p}{p^*}} + o(1)$$

by Sobolev inequality. In this way from (1.10) we get

$$S(p)\Bigg(\int\limits_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*}\,d\xi\Bigg)^{\frac{p}{p^*}} \leqslant S_{a,p^*}(p)\Bigg(\int\limits_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*}\,d\xi\Bigg)^{\frac{p}{p^*}} + o(1).$$

Since $S_{a,p^*}(p) < S(p)$ by assumption, this implies that

$$\int\limits_K |x|^{\frac{Na}{N-p}}|u_h|^{p^*}\,d\xi \leqslant \int\limits_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*}\,d\xi = o(1),$$

that contradicts (1.9). Thus, $u_h \to u \neq 0$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$.

Finally, standard arguments imply that $u_h \to u$ strongly in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$, and therefore that u achieves $S_{a,q}(p)$. For completeness we recall the argument here. From (1.8) it follows that u solves (0.5), and in particular

$$\int\limits_{\mathbb{R}^N} |x|^a|\nabla u|^p = \int\limits_{\mathbb{R}^N} |x|^{-b_a}|u|^q \leqslant \big(S_{a,q}(p)\big)^{-\frac{q}{p}}\Bigg(\int\limits_{\mathbb{R}^N} |x|^a|\nabla u|^p\Bigg)^{\frac{q}{p}},$$

by definition of $S_{a,q}(p)$. Since $u \neq 0$, this implies that $\int_{\mathbb{R}^N} |x|^a|\nabla u|^p \geqslant (S_{a,q}(p))^{\frac{q}{q-p}}$. But then, (1.7) and the lower semicontinuity of the norm in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$ imply

$$\int\limits_{\mathbb{R}^N} |x|^a|\nabla u_h|^p = \int\limits_{\mathbb{R}^N} |x|^a|\nabla u|^p + o(1),$$

that suffices to conclude that $u_h \to u$ in the uniformly convex space $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$. □

2. Proof of Theorem 0.2

In order to prove Theorem 0.2 it suffices to show that $S_{a,p^*}(p) < S(p)$. Indeed, we claim that the following estimate holds:

$$S_{a,p^*}(p) \leqslant S(p)\frac{N-p}{N}\,\frac{N+a}{N-p-a(p-1)}. \tag{2.1}$$

Notice that the right-hand side in (2.1) is strictly increasing in a, and therefore (2.1) implies $S_{a,p^*}(p) < S(p)$ for $a < 0$. To prove (2.1) we estimate $S_{a,q}(p)$ with the map

$$U(\xi) = \left(1 + |\xi|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}},$$

that achieves the best constant $S(p)$ (see [31]). We compute

$$|\nabla U|^p = \left(\frac{N-p}{p-1}\right)^p |\xi|^{\frac{p}{p-1}} \Phi^{-N}, \quad |U|^{p^*} = \Phi^{-N}, \tag{2.2}$$

where we have set $\Phi(\xi) := 1 + |\xi|^{\frac{p}{p-1}}$. An application of the divergence theorem leads to

$$\int_{\mathbb{R}^N} |x|^a |\xi|^{\frac{p}{p-1}} \Phi^{-N} = -\frac{p-1}{p}\frac{1}{N-1}\int_{\mathbb{R}^N} |x|^a \nabla\left(\Phi^{1-N}\right)\cdot\xi = \frac{p-1}{p}\frac{N+a}{N-1}\int_{\mathbb{R}^N} |x|^a \Phi^{1-N}.$$

On the other hand,

$$\int_{\mathbb{R}^N} |x|^a |\xi|^{\frac{p}{p-1}} \Phi^{-N}\, d\xi = \int_{\mathbb{R}^N} |x|^a\left(\Phi^{1-N} - \Phi^{-N}\right) d\xi,$$

and hence

$$\int_{\mathbb{R}^N} |x|^a |\xi|^{\frac{p}{p-1}} \Phi^{-N}\, d\xi = \frac{(p-1)(N+a)}{N-p-a(p-1)}\int_{\mathbb{R}^N} |x|^a \Phi^{-N}\, d\xi.$$

Thus, from (2.2) we infer

$$\int_{\mathbb{R}^N} |x|^a |\nabla U|^p\, d\xi = \left(\frac{N-p}{p-1}\right)^p \frac{(p-1)(N+a)}{N-p-a(p-1)}\int_{\mathbb{R}^N} |x|^a \Phi^{-N}\, d\xi. \tag{2.3}$$

We can compute $S(p)$ by setting $a = 0$ in (2.3):

$$S(p) = \left(\frac{N-p}{p-1}\right)^p \frac{(p-1)N}{N-p}\left(\int_{\mathbb{R}^N} \Phi^{-N}\, d\xi\right)^{\frac{p}{N}}.$$

Therefore, (2.3) and Hölder inequality imply

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^a |\nabla U|^p &\leqslant \left(\frac{N-p}{p-1}\right)^p \frac{(p-1)(N+a)}{N-p-a(p-1)}\left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}} \Phi^{-N}\right)^{\frac{N-p}{N}}\left(\int_{\mathbb{R}^N} \Phi^{-N}\right)^{\frac{p}{N}}\\ &= S(p)\frac{N-p}{N}\frac{N+a}{N-p-a(p-1)}\left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}} U^{p^*}\, d\xi\right)^{\frac{N-p}{N}},\end{aligned}$$

and the conclusion readily follows. □

Remark 2.1. We notice that the condition $p^2 < N$ suggested in [32] to get the existence of a minimizer for $S_{p-k,p^*}(p)$ is not necessary, even if up to now we are not able to prove its sufficiency (except when $p = 2$, compare with [32]).

3. Problems on cones

In this section we extend some results already proved in [12] and in the more recent papers [7,28].

Our arguments for Theorem 0.1 can be used with no modifications to study problems on cones. Accordingly with [12], we say that a cone in $\mathbb{R}^k$ is a domain $\mathcal{C}^k \subset \mathbb{R}^k$ such that $\mu x \in \mathcal{C}^k$ for every $\mu > 0$ and for every $x \in \mathcal{C}^k$. A cone $\mathcal{C}^k$ is said to be *proper* if $0 \notin \mathcal{C}^k$. Notice that $\mathbb{R}^k$ itself is a cone, $\mathbb{R}^k \setminus \{0\}$ is a proper cone, and that $(0, +\infty)$ is a proper cone in $\mathbb{R}$. The only domains in $\mathbb{R}^N$ that are invariant with respect to dilations and translations in the y-variable are of the form $\mathcal{C}^k \times \mathbb{R}^{N-k}$, where $\mathcal{C}^k$ is a cone.

In this section we take $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a proper cone in $\mathbb{R}^N$. Notice that for any $a \in \mathbb{R}$ the Hardy inequality holds on Ω:

$$\left|\frac{k-p+a}{p}\right|^p \int_\Omega |x|^{a-p}|u|^p\,d\xi \leqslant \int_\Omega |x|^a|\nabla u|^p\,d\xi \quad \forall u \in C_c^\infty(\Omega),$$

see for example [15]. If $a \neq p-k$ the Hardy constant is positive, and we can define the Banach space $\mathcal{D}_0^{1,p}(\Omega; |x|^a\,d\xi)$ by completing $C_c^\infty(\Omega)$ with respect to the norm $\int_\Omega |x|^a|\nabla u|^p$. Notice that $\mathcal{D}_0^{1,p}(\Omega; |x|^a\,d\xi) \subset \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$ for $a > (p-N)\frac{k}{N}$. By a density argument based on the Hardy inequality one can prove that $\mathcal{D}_0^{1,p}((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}; |x|^a\,d\xi) = \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$ if and only if $a > p-k$.

We start with a Sobolev-type inequality on cones.

Lemma 3.1. *Assume $p \in (1, N)$, $q \in [p, p^*]$, $a \neq p-k$ and set $b_a = N - q\frac{N-p+a}{p}$. Let $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a proper cone in $\mathbb{R}^k$. Then there exists a constant $c > 0$ such that*

$$c\left(\int_\Omega |x|^{-b_a}|u|^q\,d\xi\right)^{p/q} \leqslant \int_\Omega |x|^a|\nabla u|^p\,d\xi, \quad \forall u \in \mathcal{D}_0^{1,p}\big(\Omega; |x|^a\,d\xi\big). \tag{3.1}$$

Proof. Notice that (3.1) is the Hardy inequality if $q = p$. We are going to prove (3.1) for $q = p^*$. Fix any map $u \in C_c^\infty(\Omega)$, and set $Lu := |x|^{\frac{a}{p}}u \in C_c^\infty(\Omega)$. Notice that

$$\int_\Omega |\nabla Lu|^p\,d\xi \leqslant c\left(\int_\Omega |x|^a|\nabla u|^p\,d\xi + \int_\Omega |x|^{a-p}|u|^p\,d\xi\right) \leqslant c\int_\Omega |x|^a|\nabla u|^p\,d\xi$$

where the constants c do not depend on u. Thus,

$$\left(\int_\Omega |x|^{\frac{Na}{N-p}}|u|^{p^*}\,d\xi\right)^{\frac{p}{p^*}} = \left(\int_\Omega |Lu|^{p^*}\,d\xi\right)^{\frac{p}{p^*}} \leqslant c\int_\Omega |x|^a|\nabla u|^p\,d\xi$$

by Sobolev inequality. Then (3.1) for $q = p^*$ follows by a density argument. For $q \in (p, p^*)$ inequality (3.1) can be proved by interpolating between the cases $q = p$ and $q = p^*$, via Hölder inequality. □

Thanks to Lemma 3.1, for any $a \neq p-k$, $q \in (p, p^*]$ the infimum

$$S_{a,q}(p;\Omega) := \inf_{\mathcal{D}_0^{1,p}(\Omega;|x|^a\,d\xi)} \frac{\int_\Omega |x|^a|\nabla u|^p\,d\xi}{(\int_\Omega |x|^{-b_a}|u|^q\,d\xi)^{p/q}}$$

is positive. Notice that $S_{a,q}(p;\Omega) \geqslant S_{a,q}(p)$ if (0.1) is satisfied. One can argue as for Theorem 0.1 to prove the next result. We omit the details.

Theorem 3.2. *Let $p \in (1, N)$, $q \in (p, p^*]$, $a \neq p-k$ and let $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a proper cone in $\mathbb{R}^k$. Then $S_{a,q}(p;\Omega)$ is achieved provided that*

$$q < p^* \quad \text{or} \quad q = p^* \quad \text{and} \quad S_{a,p^*}(p;\Omega) < S(p).$$

Remark 3.3. As usual, the case $q = p^*$ is more difficult, and up to now we are not able to give general sufficient conditions for the strict inequality $S_{a,p^*}(p;\Omega) < S(p)$.

Something can be said in case $k = 1$, $p = 2$, $a \neq 1$ and $\Omega = \mathbb{R}^N_+ := (0, +\infty) \times \mathbb{R}^{N-1}$. Indeed the following facts hold:

(i) $S_{a,2^*}(2;\mathbb{R}^N_+) = S_{2-a,2^*}(2;\mathbb{R}^N_+)$ for any a (use for example Proposition B.5 in [28]);
(ii) $S_{a,2^*}(2;\mathbb{R}^N_+) = S_{a,2^*}(2;\mathbb{R}^N)$ if $a > 1$ (compare with Lemma A.7 in Appendix A).

Therefore, from Proposition A.10 we infer that if $N \geqslant 4$ then $S_{a,2^*}(2;\mathbb{R}^N_+) < S$ if and only if $0 < a < 2$. On the contrary, for any a it turns out that $S_{a,6}(2;\mathbb{R}^3_+) = S$ and it is never achieved.

Remark 3.4. Let u be an extremal for $S_{a,q}(p;\Omega)$. Then u is a nonnegative solution to (0.5) on Ω. For example, set $x = (x_1,\dots,x_k) \in \mathbb{R}^k$ and take $\Omega = (0,+\infty) \times \mathbb{R}^{k-1} \times \mathbb{R}^{N-k}$. Thus, Ω is a half-space whose boundary contains the singular set $\{x = 0\}$. Theorem 3.2 gives sufficient conditions for the existence of positive entire solutions to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(|x|^a|\nabla u|^{p-2}\nabla u) = |x|^{-b_a}u^{q-1} & \text{on } x_1 > 0,\\ u = 0 & \text{on } \{x_1 = 0\}. \end{cases}$$

Appendix A

In this appendix we restrict our attention on the less studied case $k < N$. We collect additional results and remarks about the Euler–Lagrange equations related to the Maz'ya inequality. Some of them are nowadays standard or already partially known, and can be found in literature (for example, in [6] for $a = 0$ and in [28,32] for $p = 2$, $a \geqslant 2 - k$). However, we are going to outline their proofs for sake of completeness.

First of all we notice that Theorems 0.1, 0.2 provide sufficient conditions for the existence of nontrivial weak entire solutions to

$$\begin{cases} -\operatorname{div}(|x|^a|\nabla u|^{p-2}\nabla u) = |x|^{-b_a}u^{q-1} & \text{in } \mathbb{R}^N,\\ u \geqslant 0. \end{cases} \tag{A.1}$$

The argument is well known, and we omit it.

Remark A.1. Let us take for simplicity $p = 2$. Notice that if $k \geqslant 2$ then $u > 0$ on $\{x \neq 0\}$, by the maximum principle. This is no longer true in general if $k = 1$ (compare with Section A.1).

Remark A.2. Assume $k \geqslant 2$ and $p = 2$. Then every extremal for $S_{a,q}(2)$ is a weak entire solution to

$$\begin{cases} -\operatorname{div}(|x|^a\nabla u) = |x|^{-b_a}u^{q-1} & \text{in } \mathbb{R}^N,\\ u > 0. \end{cases} \tag{A.2}$$

It has been proved in [20, Corollary 5.1], that for $a \geqslant 2 - k$ and

$$\max\left\{2, \frac{2(N-k)}{N-2+a}\right\} < q < \frac{2(N-k+1)}{N-k-1},$$

problem (A.2) has a positive entire solution which is radially symmetric in the x-variable. Notice that $\frac{2(N-k+1)}{N-k-1} > 2^*$. Actually, we can get existence since the "true critical exponent" on the class of symmetric maps is $\frac{2(N-k+1)}{N-k-1}$.

In addition, if $a \in [2(2-k), 0]$, every classical positive solution $u \in L^q(\mathbb{R}^N; |x|^{-b_a}\,d\xi)$ to

$$-\operatorname{div}\big(|x|^a\nabla u\big) = |x|^{-b_a}u^{q-1} \quad \text{on } \{x \neq 0\}$$

is radially symmetric in the x-variable [20, Corollary 5.2]. Finally, breaking symmetry occurs as $a \gg 0$, that is, minima of (0.4) are not symmetric in the x-variable (see [20, Corollary 5.3]).

It would be of interest to know whether entire solutions to (A.1) are radially symmetric in the x-variable or not, at least for $p < k$ and a close to $p - k$. Also, one may wonder if breaking symmetry occurs as $a \to +\infty$, when $q \in (p, p^*)$.

We point out a simple technical lemma that will be useful later-on.

Lemma A.3. *Let $a \geqslant p - k$. Then $C^\infty_c((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$ is dense in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,d\xi)$.*

Proof. Fix any map $v \in C^\infty_c(\mathbb{R}^N)$. For $\varepsilon > 0$ set

$$\varphi_\varepsilon(|x|) = \begin{cases} 0 & \text{if } |x| \leqslant \varepsilon^2,\\ \frac{\log|x|/\varepsilon^2}{|\log\varepsilon|} & \text{if } \varepsilon^2 < |x| < \varepsilon,\\ 1 & \text{if } |x| \geqslant \varepsilon. \end{cases}$$

It is clear that $\varphi_\varepsilon v \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, d\xi)$ and that $\nabla(v - \varphi_\varepsilon v) = (1-\varphi_\varepsilon)\nabla v - v\nabla\varphi_\varepsilon \to 0$ a.e. on $\mathbb{R}^N$, as $\varepsilon \to 0$. To prove that $\varphi_\varepsilon v \to v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, d\xi)$ it suffices to remark that

$$\int\limits_{\mathbb{R}^N} |x|^a |v\nabla\varphi_\varepsilon|^p\, d\xi \leqslant c_v \int\limits_{\mathbb{R}^k} |x|^a \left|\varphi_\varepsilon'\right|^p dx \leqslant c_v |\log\varepsilon|^{1-p}$$

since $a \geqslant p-k$, where the constants c_v do not depend on ε. The conclusion follows via Lebesgue's Theorem, since $|(1-\varphi_\varepsilon)\nabla v| \leqslant |\nabla v|$ on $\mathbb{R}^N$, and since $|x|^a|\nabla v|^p \in L^1(\mathbb{R}^N)$. □

Remark A.4. Set $\mathbb{R}^N_0 := (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$. Since $\mathbb{R}^k \setminus \{0\}$ is a proper cone in $\mathbb{R}^k$, then the results in Section 3 apply with $\Omega = \mathbb{R}^N_0$. In particular, $S_{a,q}(p; \mathbb{R}^N_0)$ is achieved for any $a \neq p-k$, $q \in (p, p^*)$; in the limiting case $S_{a,p^*}(p; \mathbb{R}^N_0)$ is achieved provided that $S_{a,p^*}(p; \mathbb{R}^N_0) < S(p)$. It can be easily proved via Hardy inequality that for $a > (p-N)\frac{k}{N}$,

$$\mathcal{D}^{1,p}_0\big(\mathbb{R}^N_0; |x|^a\, d\xi\big) = \mathcal{D}^{1,p}\big(\mathbb{R}^N; |x|^a\, d\xi\big) \cap L^p\big(\mathbb{R}^N; |x|^{a-p}\, d\xi\big).$$

Hence, $\mathcal{D}^{1,p}_0(\mathbb{R}^N_0; |x|^a\, d\xi)$ is a proper subspace of $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, d\xi)$ if and only if $a < p-k$.

If $p = 2$ and $S_{a,q}(2; \mathbb{R}^N_0) > S_{a,q}(2) > 0$, any extremal for $S_{a,q}(2)$ corresponds, up to a functional change, to a positive classical solution $v \in L^q(\mathbb{R}^N; |x|^{-b_0}\, d\xi)$ to

$$-\Delta v - \lambda|x|^{-2}v = |x|^{-b_0}v^{q-1} \quad \text{on } \{x \neq 0\},$$

that is (possibly) singular on the $N-k > 0$ dimensional subspace $\{x = 0\}$.

Remark A.5. In this remark we take $k \geqslant 2$. Let $\mathcal{C}^k$ be a cone, properly contained in $\mathbb{R}^k \setminus \{0\}$. Assume that (0.1) are satisfied and that $q < p^*$ or $S_{a,p^*}(p; \Omega) < S(p)$. Then both the infima $S_{a,q}(p; \Omega)$ and $S_{a,p}(p)$ are achieved. One can write down the Euler–Lagrange equations to infer that $S_{a,q}(p) < S_{a,q}(p; \Omega)$. This is no longer true if $k = 1$ and $a \geqslant p-1$, compare with Section A.1 below.

A.1. The case $k = 1$

When $k = 1$ the singular set $\{x = 0\}$ is an hyperplane that disconnects the domain into two proper cones. Let us point out an immediate corollary to Theorem 0.1.

Corollary A.6. *Let $k = 1$, $p \in (1, N)$ and $\frac{p(N-1)}{N-p} < q < p^*$. Then problem*

$$\begin{cases} -\Delta_p u = |x|^{-b_a}u^{q-1} & \text{in } \mathbb{R}^N, \\ u \geqslant 0, \end{cases}$$

has a weak entire ground state solution.

As observed in [20], in case $p = 2$ the solution of Corollary A.6 is even in the x-variable and decreasing for $x > 0$. In particular, u can never vanish on $\mathbb{R}^N$. This remark and the next lemma underline the contrast between the cases $a = 0 < p-1$ and $a \geqslant p-1$.

Lemma A.7. *Let $k = 1$, $p \in (1, N)$, $q \in (p, p^*]$ and $a \geqslant p-1$. Then every minimizer for $S_{a,q}(p)$ vanishes on a half-plane.*

Proof. Set $\mathbb{R}^N_- := (-\infty, 0) \times \mathbb{R}^{N-1}$ and $\mathbb{R}^N_+ := (0, +\infty) \times \mathbb{R}^{N-1}$. By Lemma A.3, there exist sequences $u_h^- \in C^\infty_c(\mathbb{R}^N_-)$ and $u_h^+ \in C^\infty_c(\mathbb{R}^N_+)$ such that $u_h^- + u_h^+ \to u$ in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, d\xi)$. Then

$$\int\limits_{\mathbb{R}^N_-} |x|^a \left|\nabla u_h^-\right|^p \to \int\limits_{\mathbb{R}^N_-} |x|^a |\nabla u|^p, \qquad \int\limits_{\mathbb{R}^N_+} |x|^a \left|\nabla u_h^+\right|^p \to \int\limits_{\mathbb{R}^N_+} |x|^a \left|\nabla u_h^+\right|^p,$$

and similarly for the weighted L^q norms. Since u^- and u^+ have disjoint supports, then

$$S_{a,q}(p) = \frac{\int_{\mathbb{R}^N} |x|^a |\nabla(u_h^- + u_h^+)|^p}{(\int_{\mathbb{R}^N} |x|^{-b} |u_h^- + u_h^+|^q)^{p/q}} + o(1)$$
$$\geqslant S_{a,q}(p) \frac{(\int_{\mathbb{R}^N_-} |x|^{-b} |u_h^-|^q)^{p/q} + (\int_{\mathbb{R}^N_+} |x|^{-b} |u_h^+|^q)^{p/q}}{(\int_{\mathbb{R}^N_-} |x|^{-b} |u_h^-|^q + \int_{\mathbb{R}^N_+} |x|^{-b} |u_h^+|^q)^{p/q}} + o(1)$$

by the Maz'ya inequality. The conclusion easily follows by letting $h \to +\infty$, since $p < q$. $\square$

By Lemma A.7 it turns out that $S_{a,q}(p; \mathbb{R}^N_+) = S_{a,q}(p)$ whenever $a \geqslant p - 1$, even if both the infima are achieved. This means that the maximum principle fails in this case. We suspect that this is not longer true for a below $p - 1$, as the case $p = 2$, $a = 0$ suggests.

A.2. The limiting case $q = p^*$

We first point out a first consequence of the action of translations in the x-variable in the limiting case $q = p^*$.

Proposition A.8. *Let $1 \leqslant k \leqslant N$, $1 < p < N$ and $a > (p - N)\frac{k}{N}$. Then $S_{a,p^*}(p) \leqslant S(p)$.*

Proof. Fix $u \in C_c^\infty(B_1^N(0))$ and $\varepsilon > 0$. Fix a point $x_0 \in \mathbb{R}^k$ with $|x_0| = 1$ and set $\xi_0 = (x_0, 0)$. Set $u_\varepsilon(\xi) := u(\varepsilon^{-1}(\xi - \xi_0)) \in C_c^\infty(B_\varepsilon^N(\xi_0))$. We estimate

$$S_{a,p^*}(p) \leqslant \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u_\varepsilon|^p \, d\xi}{(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}} |u_\varepsilon|^{p^*} \, d\xi)^{p/q}} \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{|a|} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, d\xi}{(\int_{\mathbb{R}^N} |u|^{p^*} \, d\xi)^{p/q}},$$

that is,

$$S_{a,p^*}(p) \leqslant \inf_{C_c^\infty(B_1^N(0))} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, d\xi}{(\int_{\mathbb{R}^N} |u|^{p^*} \, d\xi)^{p/q}} = S(p),$$

by the invariance of the ratio $(\int_{\mathbb{R}^N} |\nabla u|^p)(\int_{\mathbb{R}^N} |u|^{p^*})^{-p/q}$ with respect to dilations. $\square$

Remark A.9. Quite reasonably it happens that $S_{a,p^*}(p) = S(p)$ for a large enough. On the other hand, one might suspect that $S_{a,p^*}(p) < S(p)$ for a close to $p - k$ and $k < p \leqslant \sqrt{N}$. This is the case when $p = 2$ (see [14] for $k = N$ and compare with Proposition A.10 below for $k < N$).

In case $p = 2$ we can improve Theorem 0.2. In order to simplify the notation we set $S_a := S_{a,2^*}(2)$ and $S := S(2)$, so that $S_a \leqslant S$ by Proposition A.8.

Proposition A.10. *Let $1 \leqslant k < N$, $1 < p < N$ and $a > (2 - N)\frac{k}{N}$.*

1. *Let $k \geqslant 2$. Then S_a is achieved if and only if $a \leqslant 0$.*
2. *Let $k = 1$ and $N \geqslant 4$. Then S_a is achieved if and only if $a < 2$.*
3. *Let $k = 1$ and $N = 3$. Then S_a is achieved if $a \leqslant 0$ and it is not achieved if $a \geqslant 1$.*

Proof of 1. Immediate, from Theorem 0.2 and from Theorem B.5 of [28].

Proof of 2. Necessity follows again from [28, Theorem B.5]. Sufficiency for $a < 0$ follows from Theorem 0.2. Thus, we only have to show that S_a is achieved if $a \in (0, 2)$, This fact was already noticed in [32] for $a = 1$; we outline here a simpler proof for completeness. To prove that $S_a < S$ for $a \in (0, 2)$ fix $r, R > 0$ and take any bounded domain $\Omega \subset (r, R) \times \mathbb{R}^{N-1}$. Fix any map $v \in C_c^\infty(\Omega)$. Then the integration by parts implies that

$$\int\limits_{\mathbb{R}^N} |x|^a \left|\nabla\left(x^{-a/2} v\right)\right|^2 = \int\limits_{\Omega} |\nabla v|^2 - \frac{a(2-a)}{4} \int\limits_{\Omega} x^{-2} v^2 \leqslant \int\limits_{\Omega} |\nabla v|^2 - \frac{a(2-a)}{4R^2} \int\limits_{\Omega} v^2.$$

Thus

$$S_{a,2^*}(2) \leqslant \inf_{C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla v|^2 - \frac{a(2-a)}{4R^2} \int_\Omega v^2}{(\int_\Omega |v|^{2^*})^{\frac{2}{2^*}}} < S$$

since $N \geqslant 4$, by a well-known result by Brezis and Nirenberg [10, Lemma 1.1].

Proof of 3. By Theorem 0.2, S_a is achieved provided that $a < 0$. By contradiction, assume that for some $a \geqslant 1$ the infimum S_a is achieved by a map $u \in \mathcal{D}^{1,2}(\mathbb{R}^3; |x|^a\, d\xi)$. By Lemma A.7 we can assume that u is a positive entire solution to

$$\begin{cases} -\operatorname{div}(|x|^a \nabla u) = |x|^{-b_a} u^5 & \text{in } (0, +\infty) \times \mathbb{R}^2, \\ u = 0 & \text{on } \{0\} \times \mathbb{R}^2. \end{cases}$$

Set $v := x^{a/2} u$. Then Lemma A.3 and direct computations imply that v is an entire positive solution (in the sense of [26]) to

$$\begin{cases} -\Delta v = \frac{a(2-a)}{4} |x|^{-2} v + v^5 & \text{on } (0, +\infty) \times \mathbb{R}^2, \\ v = 0 & \text{on } \{0\} \times \mathbb{R}^2. \end{cases}$$

This contradicts the nonexistence result in [26, Section 6] (see also [8] for $a = 1$). □

Up to now we do not know whether S_a is achieved or not if $k = 1$, $N = 3$ and $a \in (0, 1)$. This is the only case left open.

Theorems 0.1, 0.2 and Proposition A.10 give an alternative proof of a result by Tertikas and Tintarev [32] in case $a = 2 - k$ and $p = 2$. However, as noticed in Section 6 of [32], if in addition $k > 2$ one can use the symmetry of minima to (0.4) (compare with Remark A.2) and a functional change to reduce the proof of existence in this case to an existence result by Badiale and Tarantello [6].

References

[1] B. Abdellaoui, E. Colorado, I. Peral, Some critical quasilinear elliptic problems with mixed Dirichlet–Neumann boundary conditions: Relation with Sobolev and Hardy–Sobolev optimal constants, J. Math. Anal. Appl. 332 (2007) 1165–1188.

[2] B. Abdellaoui, I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian with a critical potential, Ann. Mat. 182 (2003) 247–270.

[3] B. Abdellaoui, V. Felli, I. Peral, Existence and multiplicity for perturbations of an equation involving Hardy inequality and critical Sobolev exponent in the whole $\mathbb{R}^N$, Adv. Differential Equations 9 (2004) 481–508.

[4] B. Abdellaoui, V. Felli, I. Peral, Perturbed elliptic equations of Caffarelli–Kohn–Nirenberg type, Rev. Mat. Complut. 18 (2) (2005) 339–351.

[5] B. Abdellaoui, V. Felli, I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 9 (2006) 445–484.

[6] M. Badiale, G. Tarantello, A Sobolev–Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics, Arch. Ration. Mech. Anal. 163 (2002) 252–293.

[7] T. Bartsch, S. Peng, Z. Zhang, Existence and non-existence of solutions to elliptic equations related to the Caffarelli–Kohn–Nirenberg inequalities, Calc. Var. Partial Differential Equations 30 (2007) 113–136.

[8] R.D. Benguria, R.L. Frank, M. Loss, The sharp constant in the Hardy–Sobolev–Maz'ya inequality in the three dimensional upper half-space, preprint, arXiv:0705.3833v1.

[9] H. Brezis, L. Dupaigne, A. Tesei, On a semilinear elliptic equation with inverse-square potential, Selecta Math. (N.S.) 11 (2005) 1–7.

[10] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, Comm. Pure Appl. Math. 36 (1983) 437–477.

[11] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weight, Compos. Math. 53 (1984) 259–275.

[12] P. Caldiroli, R. Musina, On the existence of extremal functions for a weighted Sobolev embedding with critical exponent, Calc. Var. Partial Differential Equations 8 (1999) 365–387.

[13] D. Castorina, I. Fabbri, G. Mancini, K. Sandeep, Hardy–Sobolev inequalities, hyperbolic symmetry and the Webster scalar curvature problem, preprint, 2007.

[14] F. Catrina, Z.Q. Wang, On the Caffarelli–Kohn–Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math. 54 (2001) 229–258.

[15] L. D'Ambrosio, Hardy type inequalities related to degenerate elliptic differential operators, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005) 451–486.

[16] V. Felli, M. Schneider, Perturbation results of critical elliptic equations of Caffarelli–Kohn–Nirenberg type, J. Differential Equations 191 (2003) 121–142.

[17] R. Filippucci, P. Pucci, F. Robert, On a p-Laplace equation with multiple critical nonlinearities, preprint.
[18] J. Garcia Azorero, I. Peral Alonso, Existence and nonuniqueness for the p-Laplacian: Nonlinear eigenvalues, Comm. Partial Differential Equations 12 (1987) 1389–1430.
[19] J. Garcia Azorero, I. Peral Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, Trans. Amer. Math. Soc. 323 (1991) 877–895.
[20] M. Gazzini, R. Musina, On the Hardy–Sobolev–Maz'ya inequalities: Symmetry and breaking symmetry of extremal functions, Commun. Contemp. Math., in press.
[21] N. Ghoussoub, C. Yuan, Multiple solutions for quasi-linear PDES involving the critical Hardy and Sobolev exponents, Trans. Amer. Math. Soc. 352 (2000) 5703–5743.
[22] M. Ghergu, V. Radulescu, Singular elliptic problems with lack of compactness, Ann. Mat. Pura Appl. 185 (2006) 63–79.
[23] T. Horiuchi, Best constant in weighted Sobolev inequality with weights being powers of distance from the origin, J. Inequal. Appl. 1 (1997) 275–292.
[24] G. Mancini, I. Fabbri, K. Sandeep, Classification of solutions of a critical Hardy Sobolev operator, J. Differential Equations 224 (2006) 258–276.
[25] G. Mancini, K. Sandeep, Cylindrical symmetry of extremals of a Hardy–Sobolev inequality, Ann. Mat. Pura Appl. (4) 183 (2004) 165–172.
[26] G. Mancini, K. Sandeep, On a semilinear elliptic equation in $\mathbb{H}^N$, preprint, 2007.
[27] V.G. Maz'ja, Sobolev Spaces, Springer-Verlag, Berlin, 1980.
[28] R. Musina, Ground state solutions of a critical problem involving cylindrical weights, Nonlinear Anal. 68 (2008) 209–220.
[29] D. Smets, Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities, Trans. Amer. Math. Soc. 357 (2005) 2909–2938.
[30] D. Smets, V. Radulescu, Critical singular problems on infinite cones, Nonlinear Anal. 54 (2003) 1153–1164.
[31] G. Talenti, Best constants in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[32] A. Tertikas, K. Tintarev, On existence of minimizers for the Hardy–Sobolev–Maz'ya inequality, Ann. Mat. Pura Appl. 186 (2007) 645–662.
[33] K. Tintarev, Singular semilinear elliptic equations in the half-space, Rend. Istit. Mat. Univ. Trieste 33 (2001) 1–11.
[34] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, Adv. Differential Equations 2 (1996) 241–264.